Hamburg Lectures on Spectral Networks Lecture 2

I didn't get to give the lecture....

Lecture 2: Defects, Gluing, and BPS States

1. An Important Special Case: Linear Conformal Quivers
2. General Description of Punctures
3. Gaiotto Gluing
4. Other Defect "operators"
5. BPS States: General Remarks
6. A Zoo Of Class S BPS States
7. Semiclassical Description of 4d BPS States & Hyperkähler Geometry

1. An Important Special Case

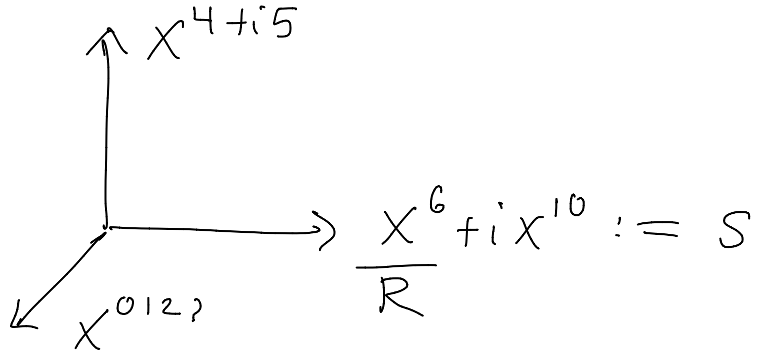
An important special case of the general discussion from the previous lecture is M-Theory on

$$\mathbb{M}^{1,3} \times C \times \mathbb{C} \times \mathbb{R}^3$$

$$x^{0,1,2,3} \qquad x^6 + i x^{10} \qquad x^4 + i x^5 \qquad x^{7,8,9}$$

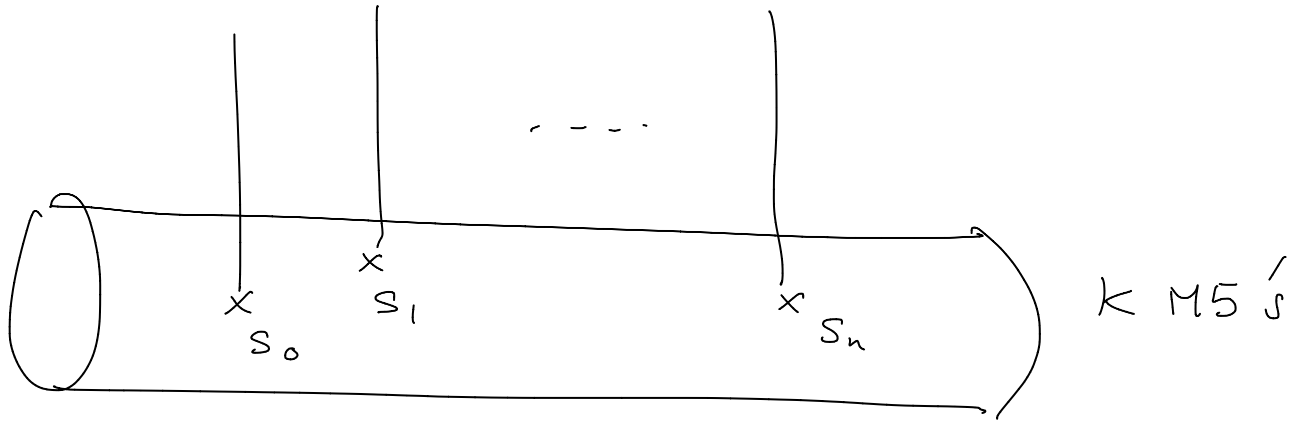
$$x^{10} \sim x^{10} + 2\pi$$

$$ds^2 = dx^\mu dx_\mu + \left[(dx^6)^2 + R^2 (dx^{10})^2 \right] + \sum (dx^i)^2$$



all @ $x^{7,8,9} = 0$

We follow an important paper of Witten: hep-th/9703166 and consider a system of k M5 branes with $W_6 = \mathbb{M}^{1,3} \times C$ at $x^{4,5,7,8,9} = 0$ together with $(n+1)$ transverse singly-wrapped M5's.



M5 wraps a holomorphic curve:

$$v^k \prod_{\alpha=0}^{n} (t - t_\alpha)$$

$$t = e^{-(x^6 + i x^{10})} \in \mathbb{C}^* \qquad v = (x^4 + i x^5)/R$$

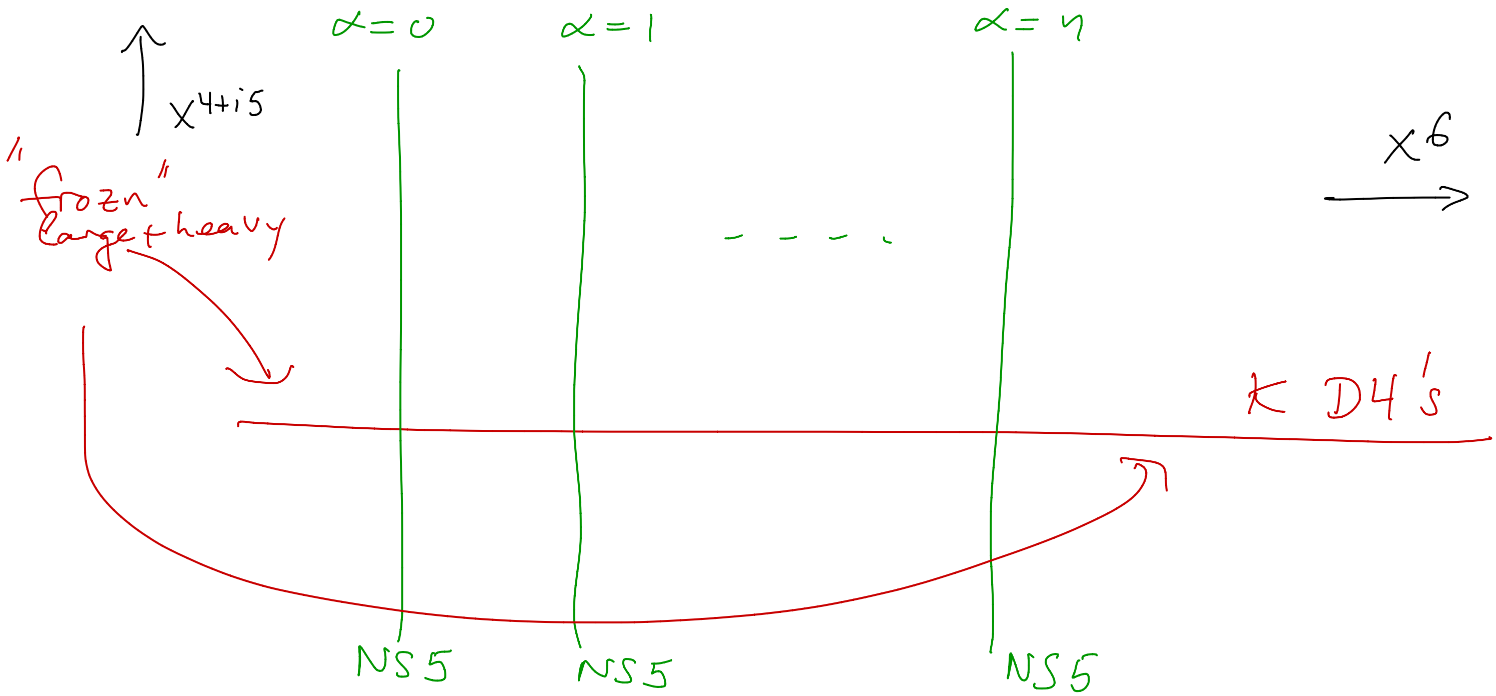
LEET is a (2,0) theory on

$$M^{1,3} \times (T^*C)_{\text{zero section}}$$

But with codimension 2 defects at $t = t_\alpha$.

$$\boxed{\text{What do these do in the 4D field theory and in the Hitchin system?}}$$

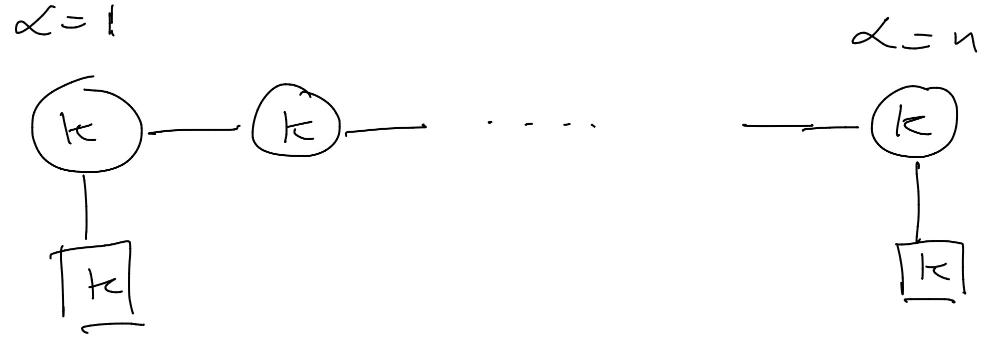
To answer this question we reduce on the M-theory circle and get a configuration of branes



At long distances $\gg (x^6_\alpha - x^6_{\alpha-1})$ The 4D effective gauge theory is a quiver gauge theory:

- In each interval $[x^6_\alpha, x^6_{\alpha+1}]$ the D4's give a $U(k)$ gauge theory: See below.
- There are bifundamental hypermultiplets across each NS5
- k fundamentals from strings @ end

Standard notation:



The Coulomb branch vacua of this theory is defined by deformations of the above singular curve that do not change asymptotics @ ∞

$$\Sigma = \{(t,v) \mid F(t,v) = 0\}$$

$$\subset \mathbb{C}_{6+i10} \times \mathbb{C}_{4+i5} \cong T^*C$$

$$F(t,v) = v^k \prod_{\alpha=0}^{n} (t-t_\alpha) + \sum_{i=1}^{k} P_i(t) v^{k-i}$$

k roots in v = k sheets of the $U(k)_\alpha$ gauge theory as the D4's separate onto their Coulomb branches

For generic p_i just one root goes to ∞ at $t \to t_\alpha$

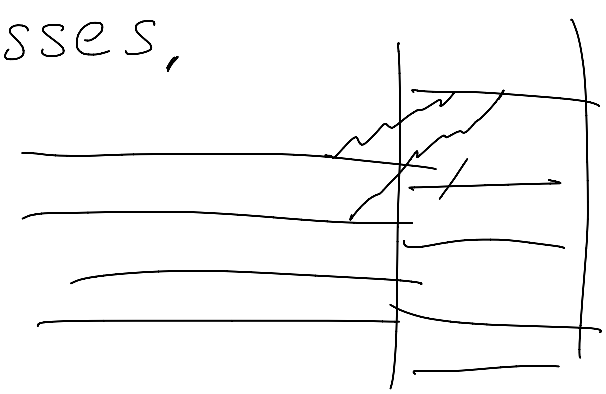
$$v_k(t) \sim \frac{1}{t-t_\alpha}\left(\frac{-p_i(t_\alpha)}{\prod_{\beta\neq\alpha}(t_\alpha - t_\beta)}\right) + \cdots$$

Also for $x^6 \longrightarrow \pm\infty \iff \overset{0}{\infty}$ we should just have k roots $\Rightarrow$ $\deg(p_i) \leq n+1$ $\Longrightarrow$ can also write

$$F(t,v) = \sum_{\alpha=0}^{n+1} t^{n+1-\alpha} q_\alpha(v) \qquad \deg(q_\alpha) = k$$

Roots of $q_0(v)$: Roots in v for $t \to \infty$

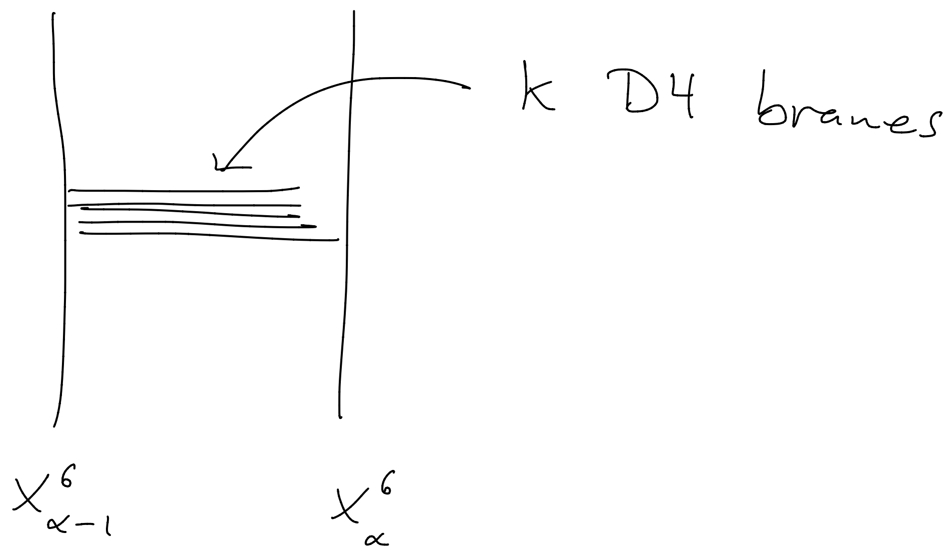
Have interpretation of fondamental string masses,



costs energy for these strings

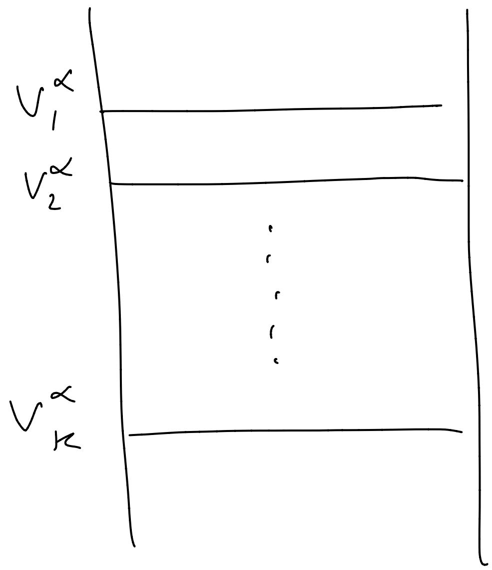
Let us understand the origin and nature of the $U(k)$ gauge theory factors better.

Focus on one interval:



In the LEET the theory on the D4's is 3+1 dimensional if we look at length scales $\gg X^6_\alpha - X^6_{\alpha-1}$. The DD strings inside each interval gives a separate $U(k)$ SYM.

The Coulomb branch is obtained by the supersymmetric brane configuration:



Now naive reduction of the action for D4 on $W_4 = \mathbb{M}^{1,3} \times [x^6_{\alpha-1}, x^6_\alpha]$ gives

Put in factors of l_{st} here

$$\frac{1}{g_{str}} \int_{\mathbb{M}^{1,3} \times [x^6_{\alpha-1}, x^6_\alpha]} \mathrm{Tr}\, F * F \longrightarrow \frac{1}{g^2_{YM}} \int_{\mathbb{M}^{1,3}} \mathrm{Tr}\, F * F$$

$$(*) \qquad \frac{1}{g^2_{YM}} = \frac{x^6_\alpha - x^6_{\alpha-1}}{g_{str}} \sim \mathrm{Re}\,(S_\alpha - S_{\alpha-1})$$

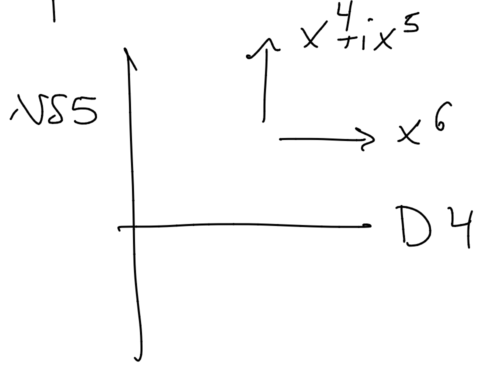
(Recall $g_{str} l_{str} = R$)

Now holomorphy dictates (Restoring factors of 2 and π we were not careful about above)

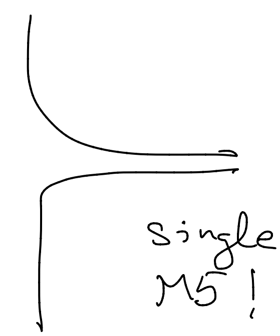
$$-i\pi\tau_\alpha = -i\pi\left(\frac{\theta_\alpha}{2\pi} + \frac{4\pi i}{g_\alpha^2}\right) = S_\alpha - S_{\alpha-1}$$

Actually this is slightly crude.

The picture



is only accurate at large scale. In fact the endpoint of the D4 defines a source for the theory on the NS5 and



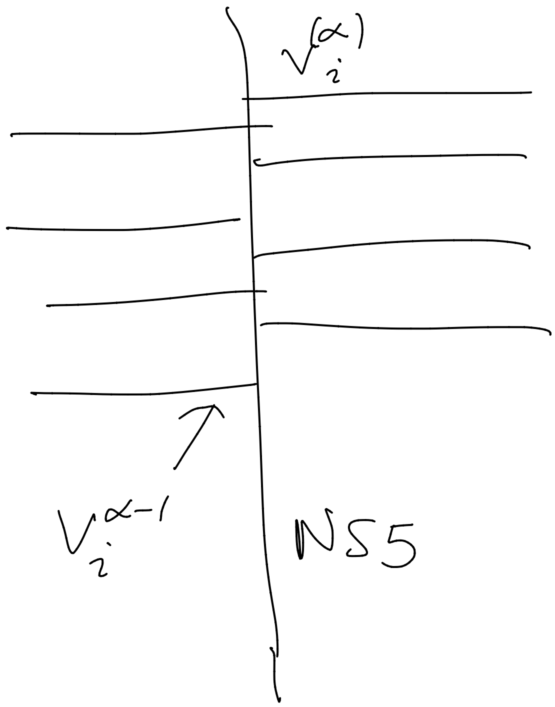
$$\nabla^2 x^6 = \delta(\cdots)$$

so

$$x^6 = \sum_{i=1}^{k} \log\left(v - v_i^{(\alpha)}\right)$$

Put together with $\dfrac{1}{g_{YM}^2} \sim \dfrac{x_\alpha^6 - x_{\alpha-1}^6}{g_{str}}$

This is a geometrization of the β-function equation. Taking into account the D4's from both sides gives:



$$x^6 = \sum_{i=1}^{k} \log(v - v_i^{\alpha}) - \sum_{i=1}^{k} \log(v - v_i^{\alpha-1})$$

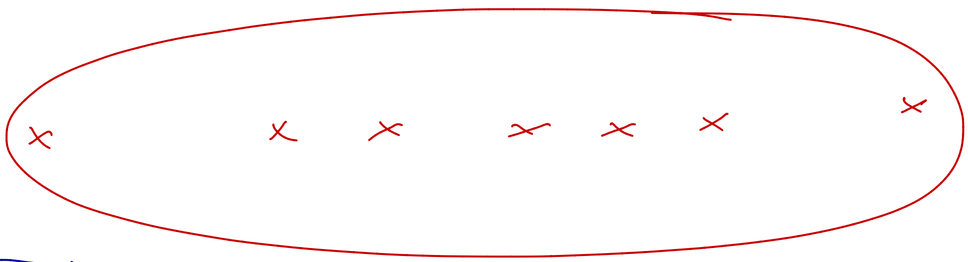
as $v \longrightarrow \infty$ the coupling approaches a finite limit and that is the way we should interpret equation (*)

In any case we should interpret the weak coupling limit $|t_\alpha / t_{\alpha-1}| \to 0$

This is a particular region of the complex structure moduli space of

$$(\mathbb{CP}^1 \setminus \{0,\infty\}) \setminus \{t_1, \ldots, t_n\}$$

Thus a weak coupling limit is associated with a particular degeneration of complex structure of



There are many ways to arrange the points $t_0, \ldots, t_n$: These will correspond to S-dual pictures of the same QFT.

Computing scalars KE $\Longrightarrow$ $\lambda = v \frac{dt}{t}$

Now from the nature of the roots of $F(t,v)$ we can deduce the behavior of the Higgs field:

$$\varphi(t) \sim \frac{dt}{t - t_\alpha} \begin{pmatrix} m_\alpha & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix} \qquad t \to t_\alpha$$

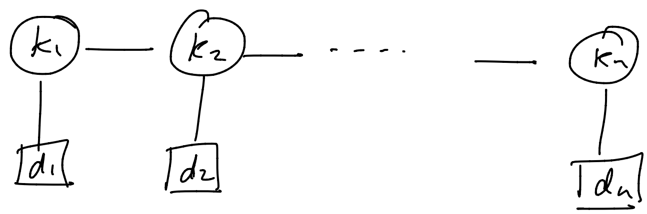
roots of $q_{n+1}(v)$

$$\varphi(t) \sim \frac{dt}{t} \begin{pmatrix} v_1^{(0)} & & \\ & \ddots & \\ & & v_k^{(0)} \end{pmatrix} \qquad t \to 0$$

roots of $q_0(v)$

$$\varphi(t) \sim \frac{dt}{t} \begin{pmatrix} v_1^{(\infty)} & & \\ & \ddots & \\ & & v_k^{(\infty)} \end{pmatrix} \qquad t \to \infty$$

We can do a similar exercise with a 4D quiver gauge theory:



β-function at α^{th} note is a positive multiple of

$$-2k_\alpha + k_{\alpha-1} + k_{\alpha+1} + d_\alpha \leq 0$$

$\uparrow$
A.F.

an interesting point here is that we must study halomorphic Curves in TN space

Get more general punctures

$$\varphi \sim \frac{r}{t - t_\alpha} dt + \cdots$$

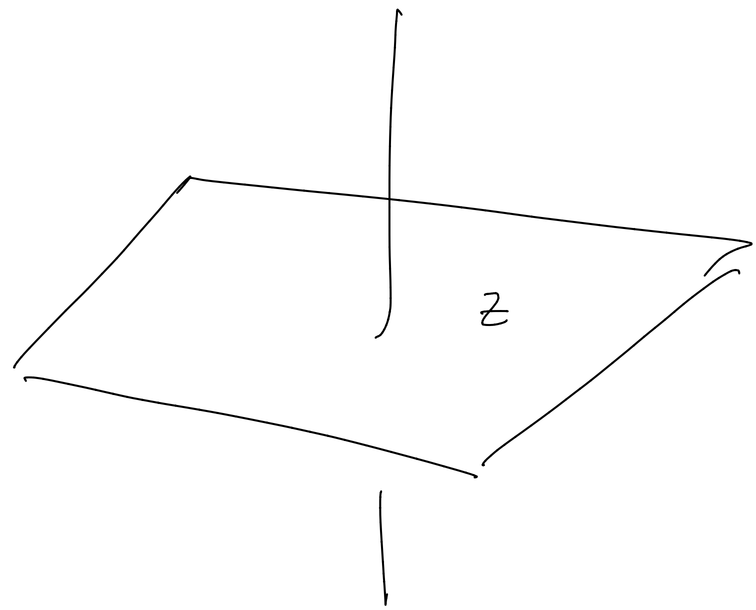
$$Z(r) = \prod_{\beta=1}^{k} U(\beta)^{l_\beta} \qquad \text{Partition of } k$$

A good way to encode this data is in terms of $\rho : sl(2) \longrightarrow sl(k)$ because that generalizes.

We also get irregular singular points (Typical for asymptotically free theories.)

2. General Description Of Punctures

More generally, it is thought that there are ½ BPS codimension two defects



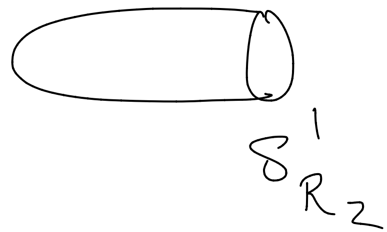
in the (2,0) theory.

So they are 4-dim'l objects that modify correlation functions — like boundary conditions. They are only characterized rather indirectly:

Recall that $S(\mathcal{C}_g)/S^1_{R_1}\times S^1_{R_2}$

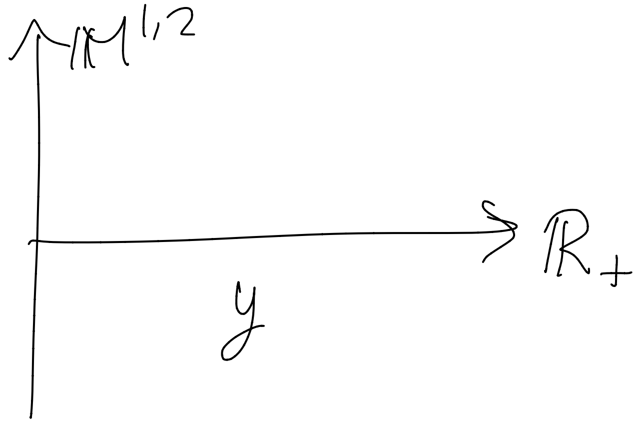
is $N=4$ $d=4$ SYM

So we consider $S[\mathfrak{g}]$ on

$\mathbb{M}^{1,2} \times S^1_{R_1} \times$ 

Reduction along $S^1_{R_2}$ is described at long distance by 5D SYM on $\mathbb{M}^{1,2} \times S^1_{R_1} \times \mathbb{R}_+$

Reducing along $S^1_{R_1}$ gives $d=4, N=4$ SYM on



The defect is "defined" by the requirement that the boundary conditions on 3 of 6 scalars are NP:

$$X^i \sim \frac{\rho(t^i)}{y} + \cdots$$

One can then argue that the induced singularity in the Hitchin system is

$$\varphi \sim \frac{r}{z} dz + \cdots$$

$r \in$ Nilpotent orbit : $\mathcal{O}_{\rho^\vee}$

Example $\mathfrak{g} = su(K)$ $\rho^\vee = \rho^\perp$

So $\rho = 0 \Longleftrightarrow [1^K]$

$\Longleftrightarrow \rho^\vee = K$

An interesting paper of Chacaltana-Distler-Tachikawa generalizes this statement to the claim that $\rho \to \rho^\vee$ is known in Lie group theory as the "Spaltenstein map."

The defect D has a global symmetry with Lie algebra $\mathfrak{f}_D$

[UPSHOT]

The general class S theory is then

$$S[\mathfrak{g}, C, D]$$

$\mathfrak{g}$ - Lie algebra with all roots $\alpha^2 = 2$

C — punctured Riemann surface

$\mathcal{D}$ — collection of defects @ punctures

For suitable defects D the theories are superconformal (in FOUR dimensions) and have a manifold of couplings (explain for math's $\mathbb{C}^*$ action on Higgs)

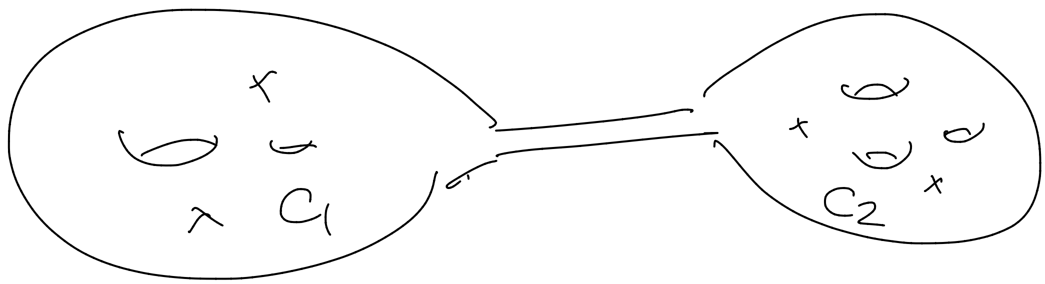
$$\mathcal{M}_{g,D} = \text{complex structures on surface with labelled punctures.}$$

Each defect contributes to global symmetry of 4D theory and

$$\bigoplus_{D_a} \mathfrak{f}_{D_a} \subset \underbrace{\text{Global Symm}(S[\mathfrak{g}, C, D])}_{\text{might be larger}}$$

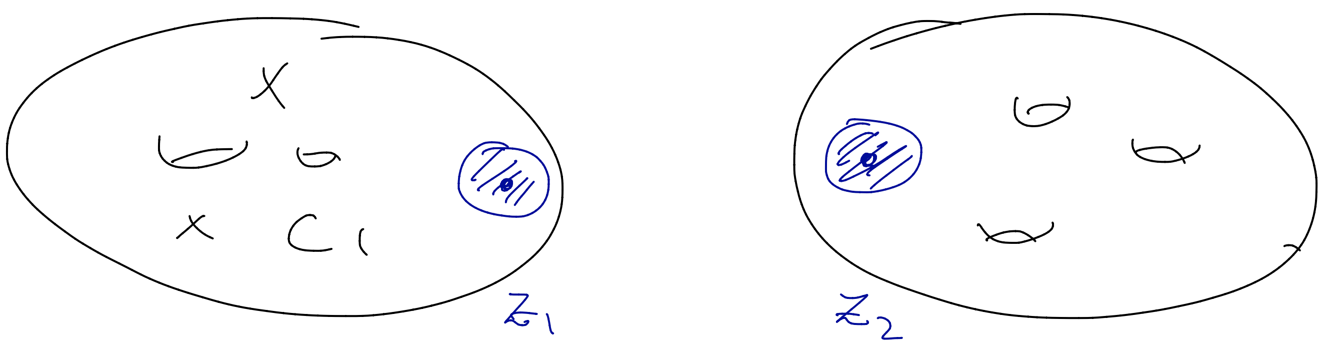
3. Gaiotto Gluing

Consider a weak coupling limit where the UV curve C degenerates



Describe this by a divisor $\mathcal{D} \subset \mathcal{M}_{g,n}$

A description of the divisor is via the plumbing construction:



Identify $z_1 z_2 = q$

q a $\perp$ coordinate to $\mathcal{D}$.

There is an elegant description of the Class $\mathcal{S}$ Theory in the limit $q \to 0$ due to D. Gaiotto. : We consider Class $\mathcal{S}$ Theories associated to C_1 and C_2 with an extra puncture at z_1 and z_2 with full $SU(K)$ global symmetry

So the 4d theory

$$\mathcal{S}[\mathfrak{g}, C_1, D_1 \cup \{D_f(z_1)\}] \times \mathcal{S}[\mathfrak{g}, C_2, D_2 \cup \{D_f(z_2)\}]$$

has a global symmetry

$$su(K) \oplus su(K) \oplus \cdots$$

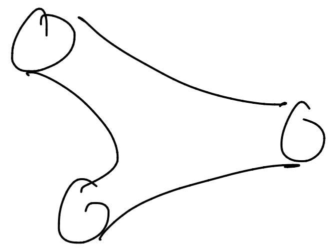
Now we gauge the diagonal $su(K)_{\text{Diag}}$ of the first two summands with gauge parameter

$$q = e^{2\pi i \tau}$$

<u>Claim:</u> This is the limiting class $\mathcal{S}$-theory

In this way one can reduce the theory to a pants decomposition:

 = VM for Lie algebra $\mathfrak{g}$

 = "trinion theory" with $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}$ global symmetry

For $\mathfrak{g} = su(2)$ the theory is a collection of hypermultiplets in the

$$2 \otimes 2 \otimes 2$$

representation. Otherwise, somewhat mysterious – related to W_N algebras in the AGT correspondence.

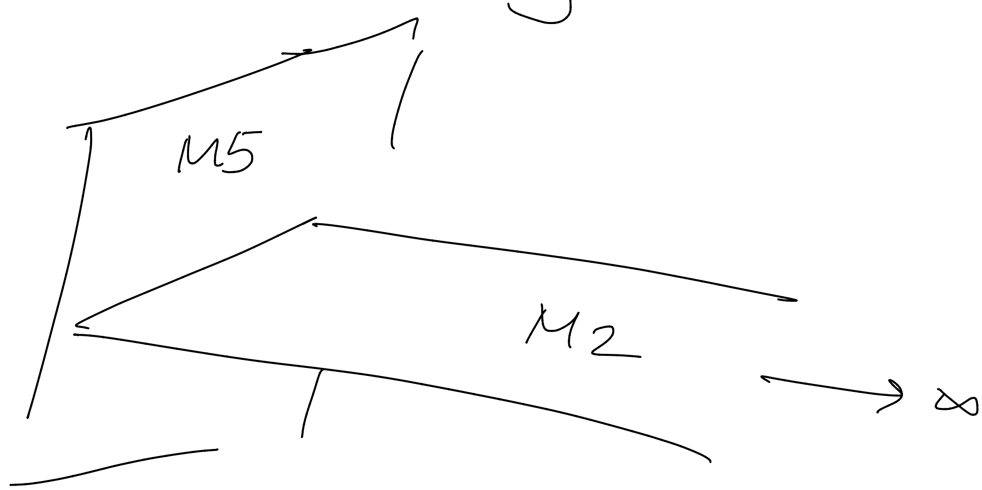
Different trinion decompositions correspond to different weak coupling limits of the theory and are related by S-duality.

The simplest case is an old observation about the geometrical interpretation of S-duality of $\mathcal{N}=4$ SYM

$$
\begin{tikzcd}
 & U(K)\,(2,0)/M^{1,3}\times S^1_{R_1}\times S^1_{R_2} \arrow[dl, "S^1_{R_1}"'] \arrow[dr, "S^1_{R_2}"] & \\
\text{5D SYM for } U(K) \text{ on } M^{1,3}\times S^1_{R_2} \arrow[d, "S^1_{R_2}"'] & & \text{5D SYM for } U(K) \text{ on } M^{1,3}\times S^1_{R_1} \arrow[d, "S^1_{R_1}"] \\
\text{4D SYM for } U(K) \text{ on } M^{1,3} \text{ with } \tau = i\dfrac{R_2}{R_1} & & \text{4D SYM for } U(K) \text{ on } M^{1,3} \text{ with } \tau = iR_1/R_2
\end{tikzcd}
$$

4. Other Defect "Operators"

The (2,0) also has a class of 2-dimensional defect "operators." In the M-Theory construction they are associated with semi-infinite M2-branes ending on the M5



In the class S context there are then a number of things we can do:

(2,0) defect dim.	Embedding in $M^{1,3} \times C$	d=4 Field Theory Interpretation
2	$S \times \{z\}$	Surface defect $\mathbb{S}_z$
2	$L \times \wp$	line defect $L(\wp)$
4	$M^{1,3} \times \{z_a\}$	D_a used to define $S[\mathfrak{g}, C, \{D_a\}]$
4	$\mathcal{H}_3 \times \wp$	domain wall
4	$S \times C$	modifies $\mathbb{S}_z$

We will focus on $\mathbb{S}_z$ and $L(\wp)$ later when describing spectral networks

5. BPS States: General Remarks

The term "BPS States" is used in physical mathematics in many different ways.

Often it has something to do with

- solitons
- magnetic monopoles
- holomorphic curves
- coherent sheaves and/or SPLAG's w/ flat v.b.
- objects in an A_∞ category

But in physics it ultimately means a "state", i.e. a positive trace class operator ρ on a Hilbert space with $\mathrm{Tr}(\rho) = 1$. It is always a pure state so ρ = Rank one projector:

$\rho = |\psi\rangle\langle\psi|$ and $\psi \in \mathcal{H}$ is in an irred. rep. of Poincaré group, an induced rep. for $SO(d) \longrightarrow \mathrm{Poin}(1,d)$ with $m^2 > 0$

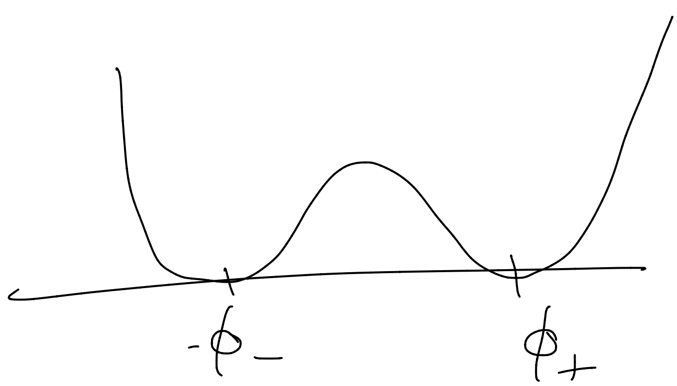
What is the relation?

Correspondence Principle

I will explain with a little digression on BPS solitons in $N=(2,2)$ field theory in 1+1 dimensions.

$$\phi: \mathbb{M}^{1,1} \longrightarrow \mathcal{X} = \mathbb{R}$$

$U(\phi) = (\phi^2 - v^2)^2$ or more generally a potential like



$$H = \int_{\mathbb{R}} dx \left((\nabla\phi)^2 + U(\phi) \right) < \infty$$

induces topology on $\mathcal{C}(\mathbb{R} \to \mathcal{X})$

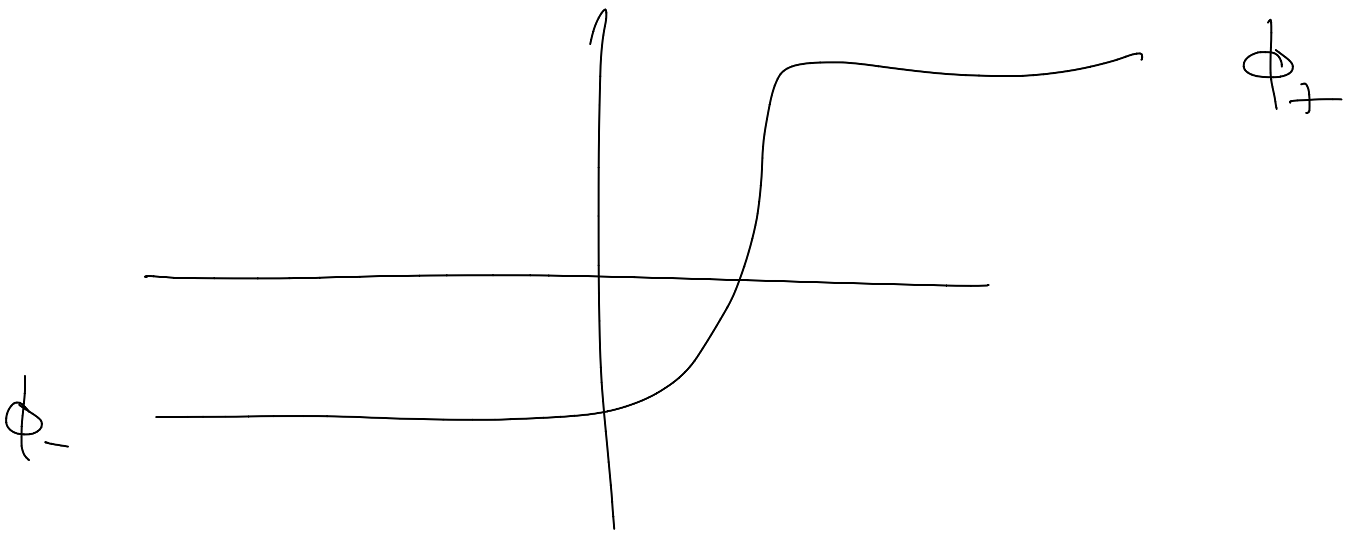
Connected components are determined by 4 boundary conditions

$$\phi(x) \xrightarrow[x \to \pm\infty]{} \text{element of } \{\phi_+, \phi_-\}$$

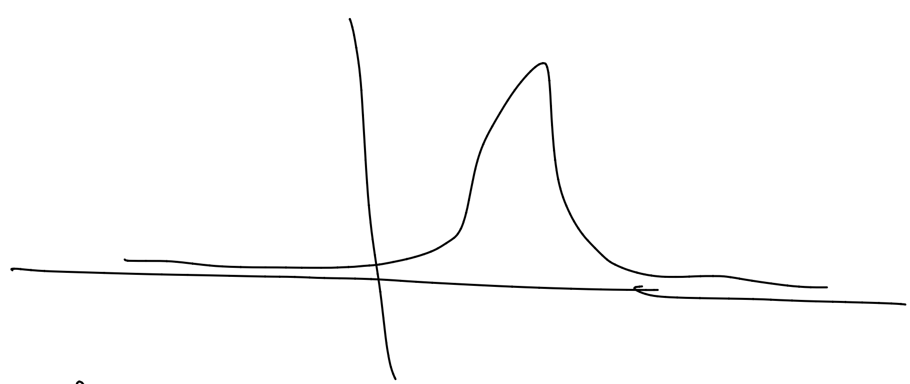
Phase space $\mathcal{P} = \coprod_{\alpha} \mathcal{P}_{\alpha}$

and we separately quantize the 4 connected components.

If $\alpha = (\phi_-, \phi_+)$ then the field looks like:



energy density for minimal energy field config. with these b.c.'s is



$\Rightarrow$ Soliton – behaves like a particle.

Now in free field theory there are "coherent states" – quantum states that correspond to well-defined classical field configurations for $\hbar \to 0$.

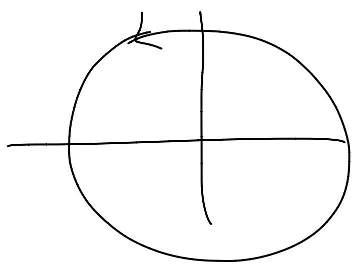
Harmonic Oscillator $\quad z \in \text{PhaseSpace} = \mathbb{R}^2$

$$\Psi_z = e^{-\frac{1}{2}|z|^2} e^{z a^\dagger} |0\rangle$$

Compute $\quad \langle \Psi_z | p(t) | \Psi_z \rangle$

$\quad \langle \Psi_z | q(t) | \Psi_z \rangle$

find



just like in classical mechanics.

In weakly coupled field theory to a classical field config. $\phi_{sol}(x)$ we try to construct a state $|\Psi_{\phi_{sol}}\rangle$ so that

$$\langle \Psi_{\phi_{sol}} | \hat{\phi}(x) | \Psi_{\phi_{sol}} \rangle = \phi_{sol}(x)$$

In general there can be important quantum corrections to this story – for example, even computing the exact energy of a coherent eigenstate of the Hamiltonian $|\Psi_{\phi_{sol}}\rangle$ is in general out of reach.

However in field theories with <u>extended</u> <u>supersymmetry</u> we can do better.

<u>Susy</u> $\{Q, Q^\dagger\} = H$

<u>Extended Susy</u> $\{Q_i, Q^{\dagger j}\} = \delta_i{}^j H$

$i,j = 1, \ldots, N$

But now $\{Q_i, Q_j\} \neq 0$ is possible

e.g. for $N=2$ $\{Q_1, Q_2\} = \hat{Z}$

$$[\hat{Z}, H] = 0$$

is a consistent susy operator algebra

It turns out that in these theories, when we quantize $\mathcal{P} = \coprod_{\alpha} \mathcal{P}_{\alpha}$ the $\hat{Z}$ operator becomes a scalar that just depends on the component α

$$\{Q_1, Q_2\} = Z_{\alpha} \cdot \mathbb{1}$$

In our soliton case α = ordered pair of classical vacua $\phi_{\pm}$

More generally α is typically a Chern class or an element of a K-theory lattice.

Write $Z_{\alpha} = e^{i\alpha} |Z_{\alpha}|$

When working out the induced rep$^{\underline{n}}$ of the super-Poincaré algebra you first quantize the Clifford algebra:

$$\{Q_i, Q^{\dagger j}\} = \delta_i^{\ j} M$$

$$\{Q_1, Q_2\} = Z_\alpha$$

Diagonalize the quadratic form.

$$\mathcal{Q}_1 = Q_1 - e^{i\varphi_\alpha} Q_2^\dagger$$

$$\mathcal{Q}_2 = Q_1 + e^{-i\varphi_\alpha} Q_2^\dagger$$

$$\{\mathcal{Q}_1, \mathcal{Q}_1^\dagger\} = 2\left(M - |Z_\alpha|\right) \Rightarrow M \geq |Z_\alpha|$$

(Bogomolnyi bound)

$$\{\mathcal{Q}_2, \mathcal{Q}_2^\dagger\} = 2\left(M + |Z_\alpha|\right)$$

$M > |Z_\alpha| \Rightarrow$ minimal Clifford rep$^{\underline{n}}$ $\mathbb{C}^{1|1} \otimes \mathbb{C}^{1|1}$

$M = |Z_\alpha| \Rightarrow$ minimal Clifford rep$^{\underline{n}}$ $\mathbb{C}^{1|1}$

Unitarity $\Rightarrow$ $\mathcal{Q}_1 = \mathcal{Q}_1^\dagger = 0$ in rep$^{\underline{n}}$

<u>Def:</u> $\mathcal{H}_{BPS} = \{\psi \mid H\psi = |Z_\alpha|\psi\}$

H, Z_α are functions of parameters
(such as $u \in \mathcal{B}$ in the Coulomb branch)

In order to count BPS states in a stable way introduce an operator ("charge") so that

$$[q, Q^{+}_{1,2}] = Q^{+}_{1,2}$$

$$\mathrm{Tr}\, x^{q} = \begin{cases} (1+x)^2 & \text{long rep } \mathbb{C}^{1|1} \hat{\otimes} \mathbb{C}^{1|1} \\ 1+x & \text{short rep.} \end{cases}$$

$$\frac{d}{dx}\Big|_{x=-1} \mathrm{Tr}\,(x^{q}) = \begin{cases} 0 \\ 1 \end{cases}$$

$$\frac{d}{dx}\Big|_{x=-1} \mathrm{Tr}_{\mathcal{H}_\alpha}\, x^{q} = \Omega(\alpha)$$

"Counts" BPS states.

6. A Zoo Of Class S BPS States

In Class S with defects there are several kinds of BPS states relevant to our story.

- 4d BPS particles

$\alpha \longrightarrow \gamma \in \Gamma_u$ = electromagnetic charge lattice

(subquotient of) $H_1(\Sigma, \mathbb{Z})$

$Z_\alpha = \oint_\gamma \lambda$

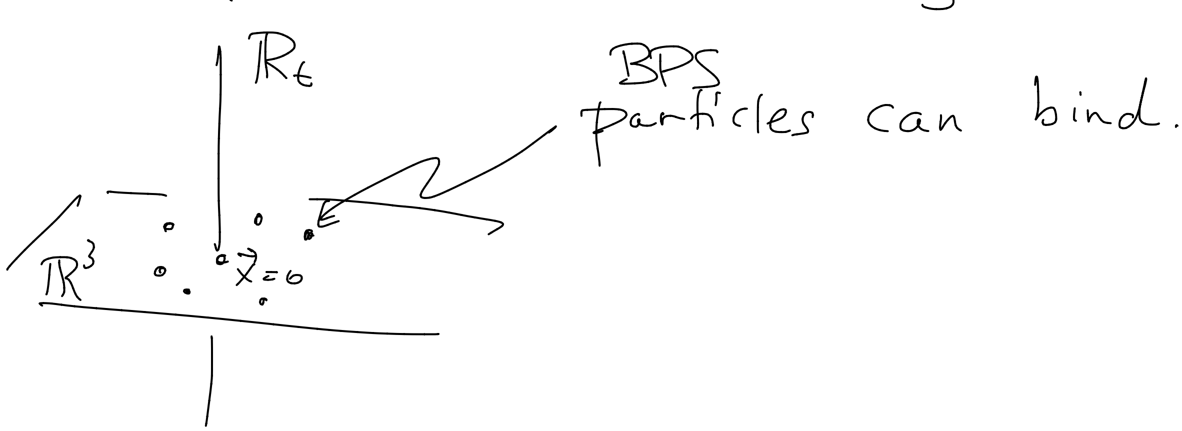
$\Omega(\gamma; u)$ piecewise constant in u

satisfies KSWCF

- 4d Framed BPS States

Defined in presence of line defect $L(p, \vartheta)$

(ϑ = phase used in defining the line defect)



$\alpha \in \Gamma_L$ = torsor for Γ

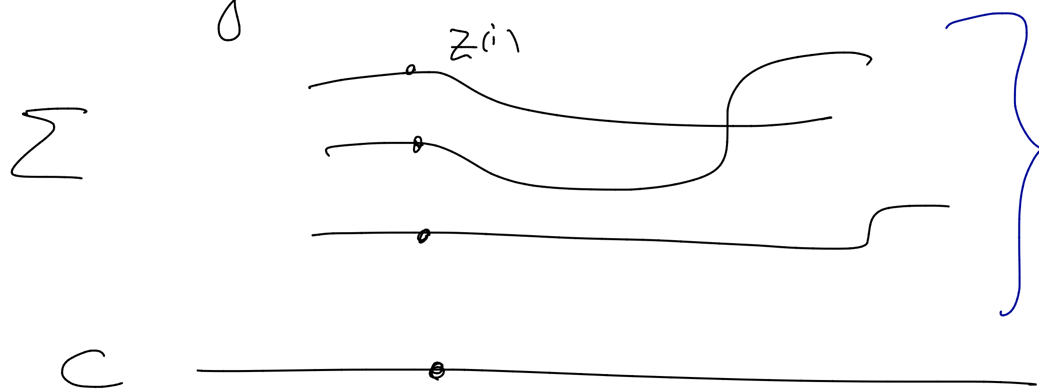
$$\overline{\underline{\Omega}}(\wp,\zeta;\cdot) : \Gamma_L \longrightarrow \mathbb{Z}$$

Satisfies a simpler WCF ($\Rightarrow$ KSWCF)

Used in Darboux expansion of vev's

$$\langle L(\wp,\zeta)\rangle = \sum_{\gamma\in\Gamma_L} \overline{\underline{\Omega}}(\wp,\zeta,\gamma)\, Y_\gamma$$

- Canonical Surface Defect Soliton Degeneracies



Pre-images $\Longleftrightarrow$ vacua for $\mathbb{S}_z$

$z \Longrightarrow \mathbb{S}_z$

$$\Gamma_{ij}(z,z) = \frac{\left\{ c \in C_1(\Sigma,\mathbb{Z}) \,\middle|\, \partial c = z^{(i)} - z^{(j)} \right\}}{c \sim c + \partial\sigma}$$

$$\Gamma(z,z) = \bigcup_{i,j} \Gamma_{ij}(z,z) \ni \alpha$$

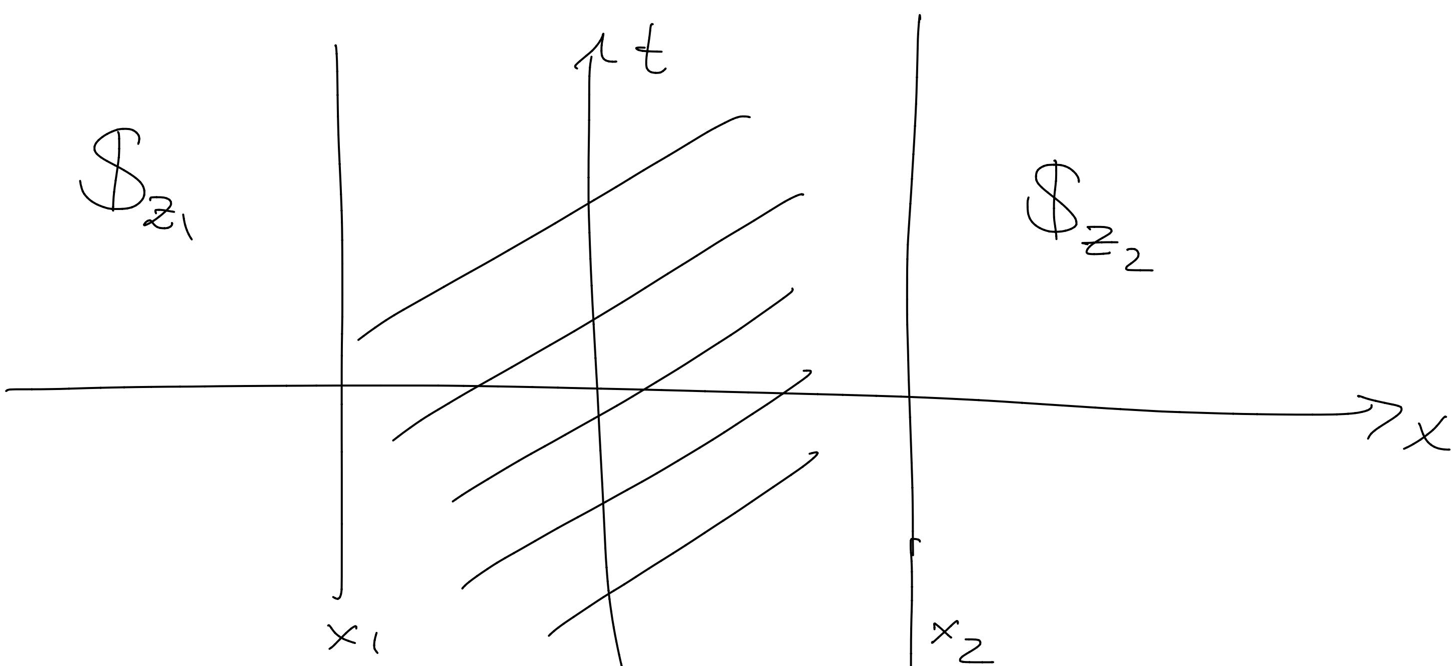
$$Z_\alpha = \oint_\alpha \lambda \sim$$ Difference of critical values of superpotential

Degeneracies $\mu(\alpha) = \mathrm{Tr}_{\mathcal{H}_\alpha} F(-1)^F$ in theory of surface defect.

- Framed Degeneracies For Interfaces

$\mathbb{S}_z$: 1+1D theory whose couplings depend on z.

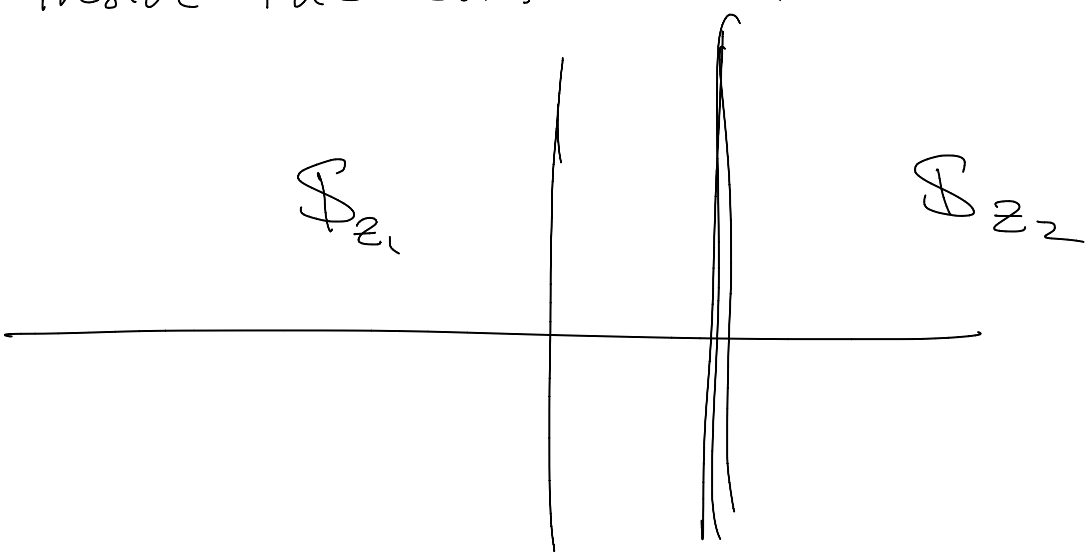
Imagine varying couplings with the 1D spatial variable:



$$z_1 \xrightarrow{z(x)} z_2$$

$z(x)$ describes a <u>path</u> in $\mathbb{C}$

At long distance we have a line defect inside the surface defect



and we have analogs of framed BPS States

$$\alpha \in \Gamma(z_1, z_2) = \bigcup_{i,j'} \Gamma_{ij'}(z_1, z_2)$$

$$\Gamma_{ij'}(z_1, z_2) = \frac{\{\tilde{\gamma} \in C_1(\Sigma, \mathbb{Z}) \mid \partial\tilde{\gamma} = z_1^{(i)} - z_2^{(j')}\}}{\tilde{\gamma} \sim \tilde{\gamma} + \partial\sigma}$$

$$Z_\alpha = \int_\alpha \lambda \quad ; \quad \underline{\overline{\Omega}}(p, \zeta, \cdot) \longrightarrow \mathbb{Z}$$

7. Semiclassical Description

There are many ways to define the BPS degeneracies. One nice way applies to Lagrangian $d=4$ $\mathcal{N}=2$ theories.
So, these are defined by the data:

G – Compact s.s. Lie group
$\mathcal{R}$ – quaternionic representation

We need to work "at infinity" in $\mathcal{B}$ in regions corresponding to weak coupling (Recall $\Omega(\gamma; u)$ is piecewise constant, jumping only on real cod. 1 walls of marginal stability.)

In these regions we have a canonical duality frame:

$$\Gamma \cong \Gamma_{mg} \oplus \Gamma_{el}.$$

$$\Gamma_{mg} \cong \Lambda_{\text{cocharacter}}(G)$$
$$\Gamma_{el} \cong \Lambda_{\text{weight}}(G)$$

$\gamma = \gamma_m \oplus \gamma_{el}$

γ_m = magnetic charge, determines a magnetic monopole moduli space

$$\mathcal{M}_{mag\,mon}(\gamma_m, X_\infty) = \{ F = * \nabla X \text{ on } \mathbb{R}^3 \}$$

$$X_\infty = Re(\zeta^{-1} a) \in \mathfrak{t}$$

asymptotic Higgs vev.

We have to write

$$\mathcal{M}_{mag\,mon} = \mathbb{R}^3 \times \frac{\mathbb{R} \times \mathcal{M}_{strong.\,cent.}}{\mathbb{Z}}$$

leading to a lot of technical headaches

Now $\mathcal{R}$ determines (via a universal construction) a hyperholomorphic bundle

$$\mathcal{E}_{\mathcal{R}} \longrightarrow \mathcal{M}_{mag.\,mon.}$$

We then consider the Dirac operator $\not{D}$ coupled to $\mathcal{E}_R \longrightarrow \mathcal{M}_{mag.mon.}$

(Actually, it is not exactly the D.O. Rather we add Clifford mult. by a hyperhalo. v.f. determined by $Y = \mathrm{Im}\,\tilde{\zeta}^{-1} a_D$.)

Suitably separating out the center of mass, the Hilbert space of BPS states with magnetic charge γ_m is just

$$\ker_{L^2} \not{D}\Big|_{\mathcal{M}_{stron.cent.}} = \left\{ \begin{array}{c} \text{BPS states with} \\ \gamma = \gamma_m \oplus * \end{array} \right\}$$

Now $T \subset G$ has a hyper-halo action on $\mathcal{M}_{strong.cent.}$, lifts to $S \otimes \mathcal{E}_R$ and commutes with the Dirac operator. The isotypical $\gamma_e \in \Lambda_{wt}(G)$ space gives

$$\mathcal{H}_{BPS}^{\gamma_m \oplus \gamma_e} \cong \left(\ker_{L^2} \not{D}\Big|_{\mathcal{M}_0} \right)^{\gamma_e}$$

This space is <u>also</u> a representation of the rotation action $SU(2)$ on $\mathbb{R}^3$

and

$$\Omega(\gamma_m \oplus \gamma_e) = \mathrm{Tr}_{\mathcal{H}_{BPS}^{\gamma_m \oplus \gamma_e}} (-1)^{2J_3}$$

There is a very similar description of framed BPS degeneracies $\overline{\Omega}$ using singular monopoles –

See my papers with

D. van den Bleeken

D. Brennan

A. Royston

for more details.